Mathematical Methods for Marine Energy Extraction

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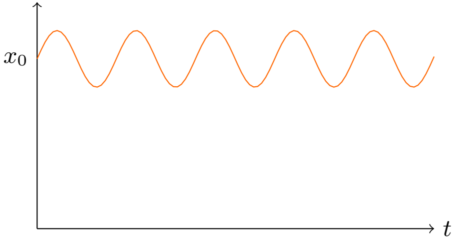
Outline

Luenberger observer

$$(1) \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \text{ unknown}, \\ y(t) = Cx(t). \end{cases}$$



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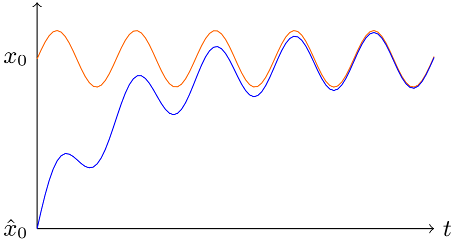

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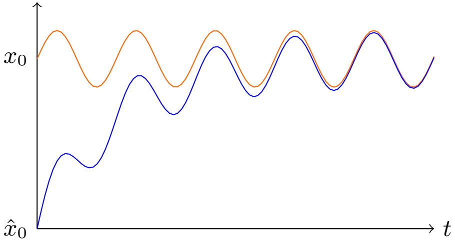

Is there a *convenient* way to choose the *observer gain* L ?

Luenberger observer

Note that

$$(2) \iff \begin{cases} \dot{\hat{x}}(t) = (A - LC)\hat{x}(t) + (Bu(t) + Ly(t)), \\ \hat{x}(0) = \hat{x}_0. \end{cases}$$

and then $x(t) - \hat{x}(t) = \mathrm{e}^{(A-LC)t}\,(x(0) - \hat{x}(0))$

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Theorem ▶ Identity observer Theorem [Luenberger]

Given a completely *observable* system (1), an identity observer of the form (2) can be constructed, and the coefficients of the characteristic polynomial of the observer can be selected arbitrarily.

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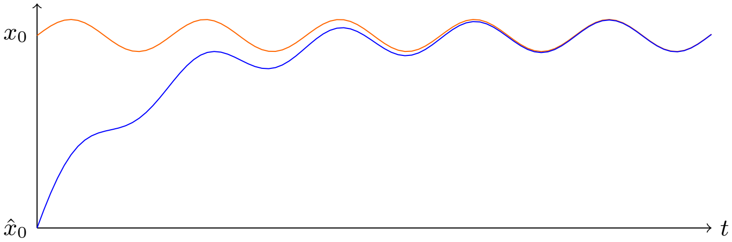
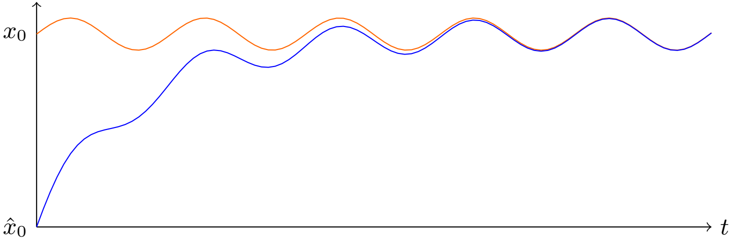
PROPOSITION 1.1

We assume System (1) is observable and the eigenvalues of $A - LC$ are negative and simple. Then, we have

$$\left\| \mathrm{e}^{(A-LC)t} \right\| \leq \gamma \mathrm{e}^{-\mu t}$$

with $\mu := \min_{\nu \in \sigma(A-LC)} |\nu|$ and $\gamma := \mathrm{cond}(V) = \left\| V^{-1} \right\| \|V\|$, where V is the matrix whose rows are the eigenvectors of $A - LC$ and $\|\cdot\|$ represents the induced 2-norm of a matrix.

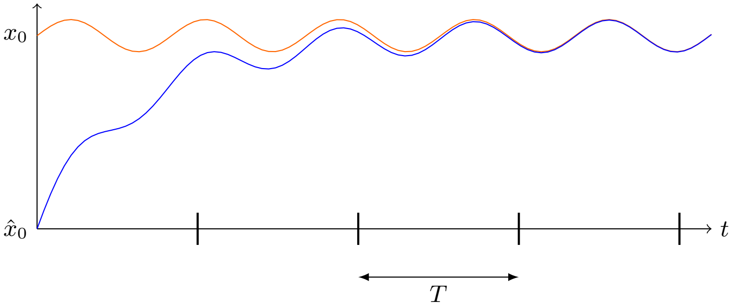

x_0
$\hat{x}_0$
t



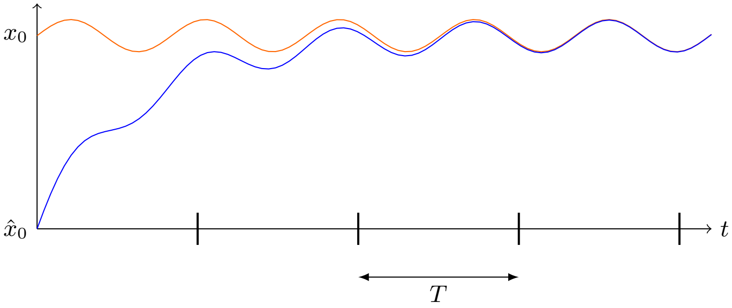
Luenberger observer

Time-parallel method

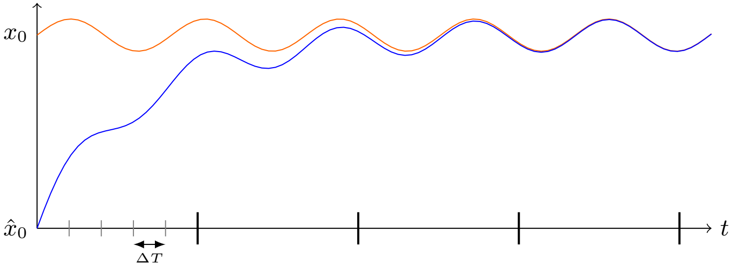
Diamond strategy



- Divide the time interval into *windows* W_ℓ of a given length $T > 0$.

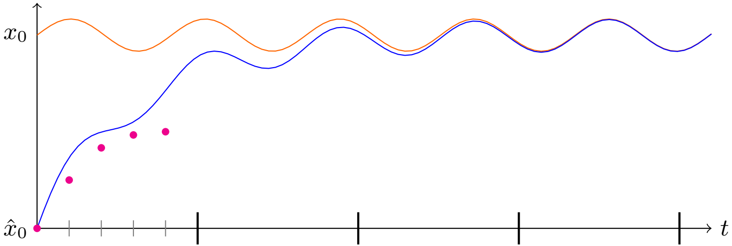


- Divide the time interval into *windows* W_ℓ of a given length $T > 0$.
- Solve Equation (2) on each window, in a sequential order, using a time-parallel algorithm.



- Decompose W_ℓ into N *subintervals* of length ΔT.

Diamond strategy (time-parallelization)



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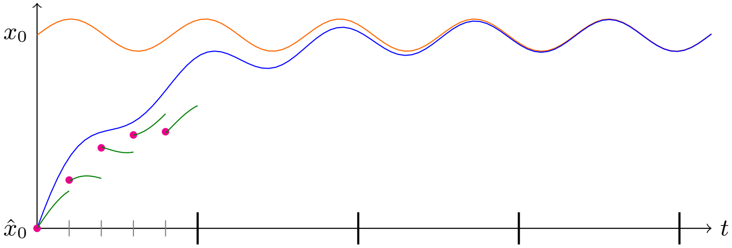

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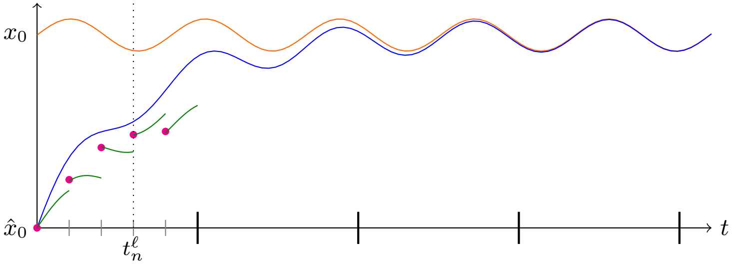
- We then construct a parallel version $\hat{x}_{\|}(t)$ of Equation (2) in each subinterval.



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- Imposing initial conditions induces discontinuities (jumps) at t^ℓ_n:

$$J^h_{\ell,n} := \hat{X}^h_{\ell,n} - \hat{x}_\parallel({t^\ell_n}^-).$$

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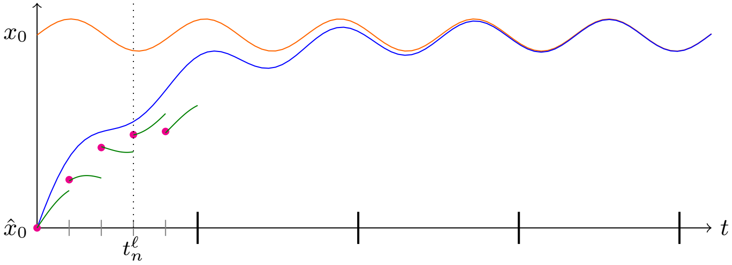

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- We then construct a parallel version $\hat{x}_\parallel(t)$ of Equation (2) in each subinterval.
- Imposing initial conditions induces discontinuities (jumps) at t_n^ℓ:

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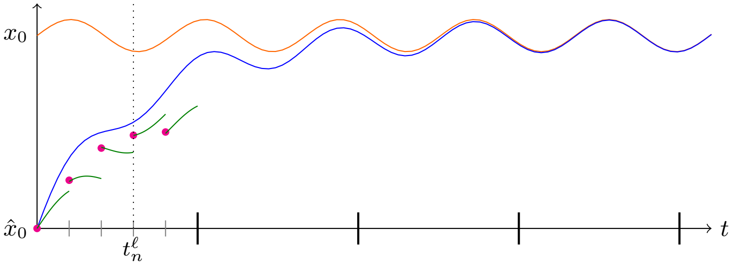
Next step : define a suitable stopping criterion !

Diamond strategy (stopping criterion)



Lemma

Under the assumptions of Proposition 1.1, we have

$$\left\|(x - \hat{x}_{\|})(t_n^{\ell\,-})\right\| \leq \gamma\Big(e^{-\mu t_n^\ell}\,\|x(0) - \hat{x}(0)\| + e^{-\mu \Delta T}\,\left\|J_{\ell,n-1}^h\right\|\Big).$$

Proposition 1.2 ▶ A posteriori estimate

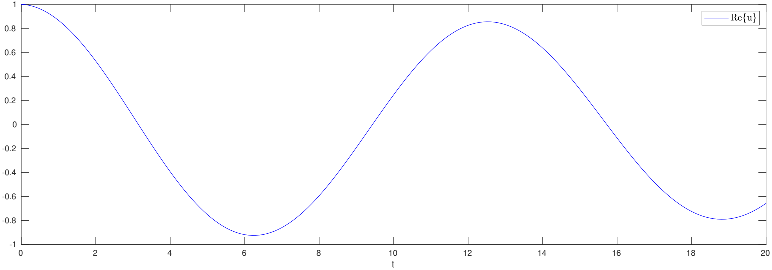
Let us assume that h is obtained from the stopping criterion in W_ℓ

$$\max_{1\leq n\leq N} \left\| J^h_{\ell,n} \right\| \leq \widetilde{\gamma} \mathrm{e}^{-\mu \ell T}$$

where $\widetilde{\gamma}$ is an arbitrary parameter. Then, the rate of convergence of $\hat{x}_{\|}(t)$ to $x(t)$ is bounded by μ, i.e.

$$\left\| (x - \hat{x}_{\|})(t_n^{\ell\,-}) \right\| \leq \gamma \mathrm{e}^{-\mu \Delta T} \left(\|x(0) - \hat{x}(0)\| + \widetilde{\gamma} \right) \mathrm{e}^{-\mu \ell T}.$$

Parareal algorithm

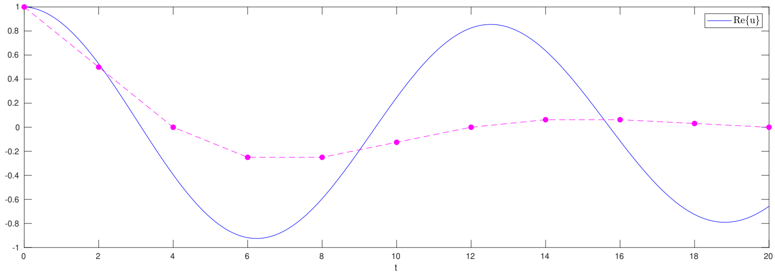


Dahlquist equation $\dot{u}(t) = -\frac{\mathrm{i}}{2}u$ in $[0, 20]$

To solve the problem

$$\begin{cases} \dot{u}(t) = f(u(t)), \quad t \in [0, T] \\ u(0) = u_0 \end{cases}$$

we decompose the time interval on N subintervals, denoted by (t_{n-1}, t_n).



- Impose arbitrary values on the subintervals by using the coarse solver $\mathcal{G}$:

$$U_0^0 = u_0,\ U_n^0 = \mathcal{G}(t_n, t_{n-1}, U_{n-1}^0).$$

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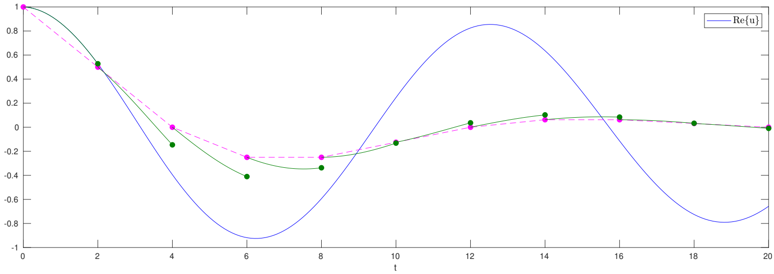

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- Using the fine solver $\mathcal{F}$, solve in parallel

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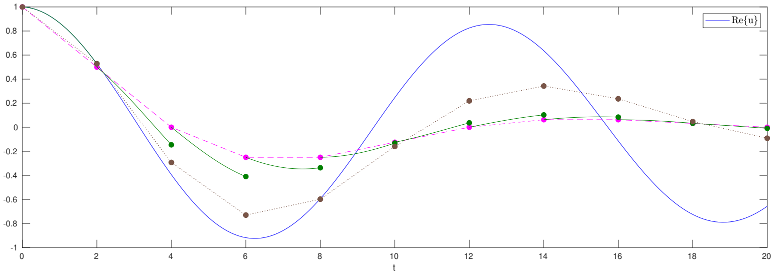

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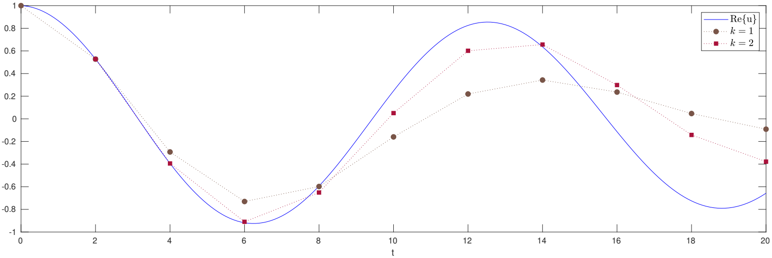
$$\begin{cases} \dot{u}(t) = f(u(t)), & t \in [t_{n-1}, t_n] \\ u(t_{n-1}) = U_{n-1}^0. \end{cases}$$

- Smooth the discontinuities previously introduced by defining

$$U_n^1 := \mathcal{F}(t_n, t_{n-1}, U_{n-1}^0) + \mathcal{G}(t_n, t_{n-1}, U_{n-1}^1) - \mathcal{G}(t_n, t_{n-1}, U_{n-1}^0).$$

Parareal algorithm



At iteration k:

- compute $\{\mathcal{F}(t_n, t_{n-1}, U_{n-1}^{k-1})\}_{n=1}^{N}$ in parallel.
- Update the sequence

$$U_n^k := \mathcal{F}(t_n, t_{n-1}, U_{n-1}^{k-1}) + \mathcal{G}(t_n, t_{n-1}, U_{n-1}^{k}) - \mathcal{G}(t_n, t_{n-1}, U_{n-1}^{k-1})$$

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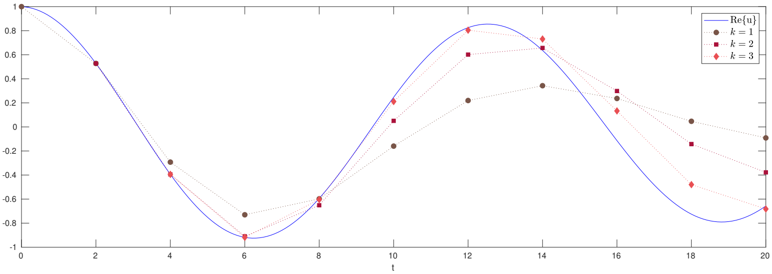

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What about its convergence ?

Theorem ▶ Convergence of Parareal [Gander and Hairer]

(...) at iteration k of the Parareal algorithm, we have the bound

$$\left\|u(t_n) - U_n^k\right\| \leq \frac{C_3}{C_1}\frac{\left(C_1 \Delta T^{p+1}\right)^{k+1}}{k!}\left(1 + C_2 \Delta T\right)^{n-(k+1)} \prod_{j=0}^{k}(n-j).$$

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- Superlinear rate of convergence.

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- Superlinear rate of convergence.
- Among other assumptions, $\mathcal{F}(t_n, t_{n-1}, U_{n-1}^k)$ is the exact solution on (t_{n-1}, t_n), and $\mathcal{G}$ must satisfy

$$\|\mathcal{G}(t + \Delta T, t, x) - \mathcal{G}(t + \Delta T, t, y)\| \leq (1 + C_2 \Delta T) \|x - y\|.$$

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- The result is well suited for non-decaying problems.

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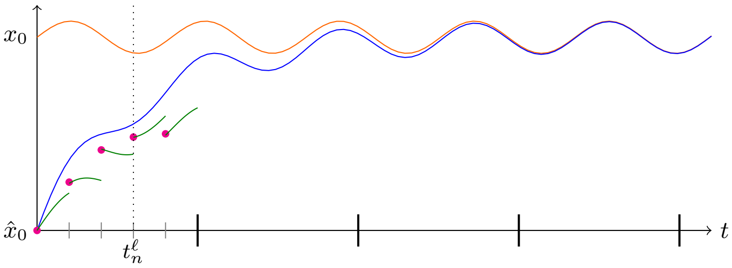
Theorem 1.3 ▶ Convergence of Parareal for decaying problems [Kwok, Riffo and Salomon]

(...) We also assume that $\mathcal{F}$ and $\mathcal{G}$ are Lipschitz with respect to the initial conditions:

$$\max \left\{\left\|\mathcal{F}(t_n, t_{n-1}, y) - \mathcal{F}(t_n, t_{n-1}, z)\right\|, \left\|\mathcal{G}(t_n, t_{n-1}, y) - \mathcal{G}(t_n, t_{n-1}, z)\right\|\right\} \leq \varepsilon \left\|y - z\right\|,$$

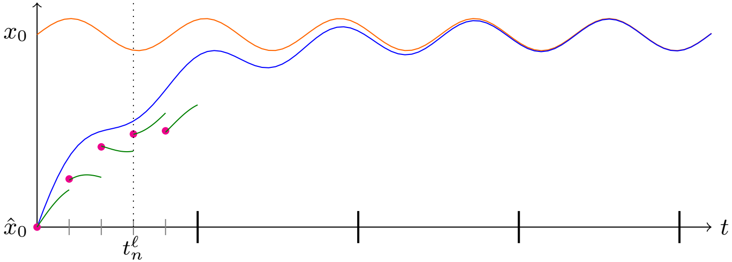
for a constant $\varepsilon \in (0, 1)$. Then, after k iterations of the Parareal algorithm, we have

$$\left\|U_n^k - u(t_n)\right\| \leq \begin{cases} 0 & n \leq k \\ \alpha \beta^k \sum_{i=0}^{n-k-1} \binom{k+i}{k} \varepsilon^i & n > k. \end{cases}$$



- The Luenberger observer $\hat{x}(t)$ is a decaying problem (Proposition 1.1).

Diamond strategy (Parareal case)



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- The number of parareal iterations $\{k_\ell\}_\ell$ can be determined from
  (a) Proposition 1.2 (*a posteriori* estimate)

$$\max_{1\leq n\leq N} \left\| \hat{X}_{\ell,n}^{k_\ell} - \hat{x}_\parallel(t_n^\ell) \right\| \leq \widetilde{\gamma}\mathrm{e}^{-\mu\ell T}, \quad \widetilde{\gamma} \text{ arbitrary.}$$

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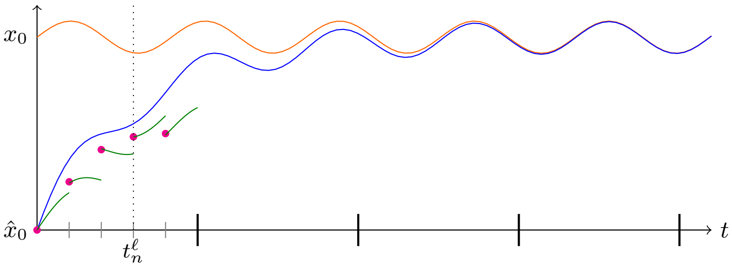

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  (b) Theorem 1.3 (*a priori* bound)

$$\left\| \hat{X}_{\ell,n}^{k_\ell} - \hat{x}_\parallel(t_n^\ell) \right\| \leq \alpha\beta^{k_\ell} \sum_{i=0}^{n-k_\ell-1} \binom{k_\ell+i}{k_\ell} \varepsilon^i, \quad n > k_\ell.$$

Diamond strategy (Parareal case)

Theorem 1.4

We keep the assumptions of Proposition 1.1 and Theorem 1.3. For a window W_ℓ and $\widetilde{\gamma} > 0$, we define

$$k_\ell = \begin{cases} \min S_\ell & S_\ell \neq \emptyset \\ k_{\ell-1} & S_\ell = \emptyset \end{cases}$$

where

$$S_\ell = \left\{ k \in \mathbb{N}^*, k \leq N-1 : \alpha \beta^k \sum_{i=0}^{N-k-1} \binom{k+i}{k} \varepsilon^i \leq \widetilde{\gamma} \mathrm{e}^{-\mu \ell T} \cdot \frac{1-\varepsilon}{\alpha(1-\varepsilon^N)} \right\}.$$

Suppose that we apply the *Diamond strategy* using k_ℓ iterations of the Parareal algorithm. Then, the stopping criterion is satisfied.

Complexity analysis (Parareal case)

We define the efficiency of the algorithm as

$$E = \frac{\tau_s}{N\tau_p}$$

where τ_s, τ_p are the CPU times required to reach a given tolerance Tol by using a sequential and parallel solver, respectively; and N represents the number of available processors.

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Theorem

The estimated efficiency of the *Diamond Strategy* is given by

$$E^{th} = \frac{\ell^* \tau^{\mathcal{F}}_{\Delta T}}{\tau^{\mathcal{F}}_{\Delta T} + N\tau^{\mathcal{G}}_{\Delta T}} \left(\sum_{\ell=0}^{\ell^*-1} k_\ell \right)^{-1},$$

where $\tau^{\mathcal{F}}_{\Delta T}$, $\tau^{\mathcal{G}}_{\Delta T}$ represents the amount of time spent in solving (2) over an interval of size ΔT with $\mathcal{F}$ and $\mathcal{G}$, respectively, and

$$\ell^* := \min \left\{ \ell \in \mathbb{N} : \left(\widetilde{\gamma}(1 + \mathrm{e}^{-\mu \Delta T}) + \gamma \mathrm{e}^{-\mu \Delta T} \, \|x(0) - \hat{x}(0)\| \right) \mathrm{e}^{-\mu \ell T} \leq \mathrm{Tol} \right\}.$$

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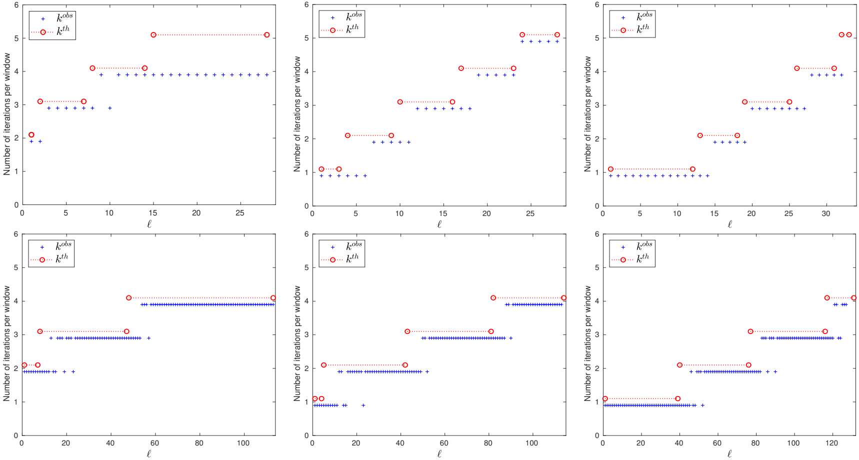
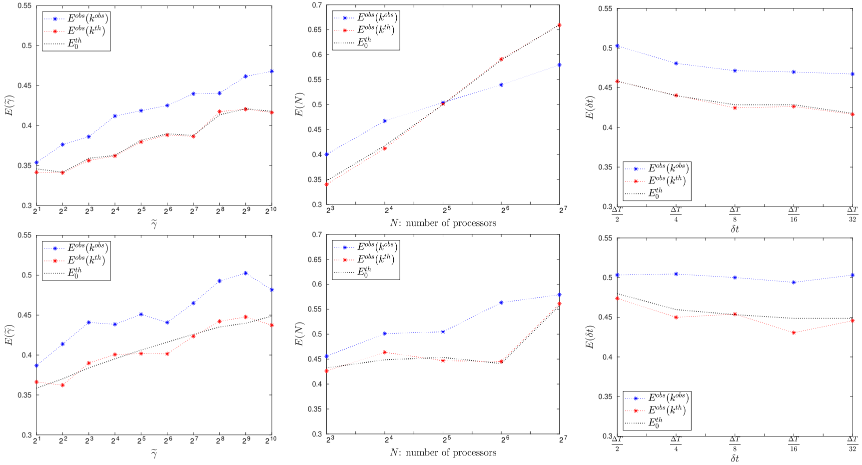


(a) $\widetilde{\gamma} = 10^{-3}$ (b) $\widetilde{\gamma} = 1.$ (c) $\widetilde{\gamma} = 10^{3}.$

Figure: Comparison between k^{th} and k^{obs}, for $N = 16$ and $\delta t = \frac{\Delta T}{2^5}$. The eigenvalues of $A - LC$ are $\{-0.8, -1\}$ (top) and $\{-0.2, -0.25\}$ (bottom).



(a) $E(\widetilde{\gamma})$, for $N = 16$ and $\delta t = \frac{\Delta T}{2^5}$.

(b) $E(N)$, for $\delta t = \frac{\Delta T}{2^5}$ and $\widetilde{\gamma} = 2^{10}$.

(c) $E(\delta t)$, for $N = 16$ and $\widetilde{\gamma} = 2^{10}$.

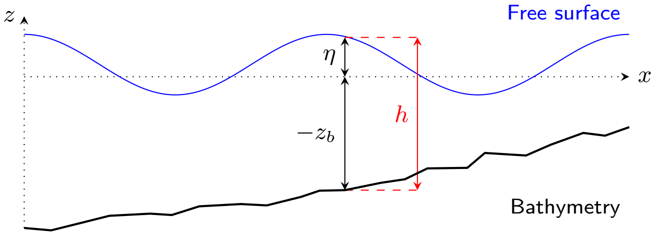
Figure: Comparison between $E^{obs}(k^{obs})$, $E^{obs}(k^{th})$ and E_0^{th}. The eigenvalues of $A - LC$ are $\{-0.8, -1\}$ (top) and $\{-0.2, -0.25\}$ (bottom).

Perspectives

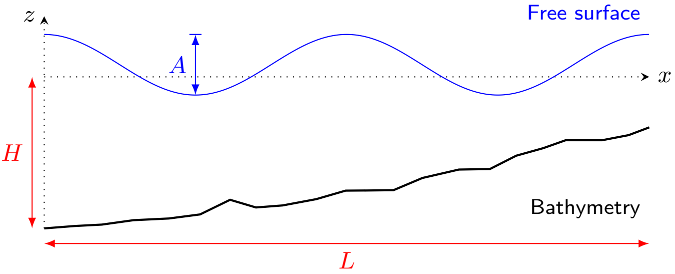
- Use of other time-parallelization algorithms (e.g. ParaExp).
- Extension to a stochastic framework (continuous Kalman filter).
- Considering a *variable* window approach.

Outline

Derivation of the wave model



- $\Omega_t = \{(x, z) \in \Omega \times \mathbb{R} \mid -z_b(x) \leq z \leq \eta(x, t)\}, \quad t \geq 0.$



- $\Omega_t = \{(x, z) \in \Omega \times \mathbb{R} \mid -z_b(x) \leq z \leq \eta(x, t)\}, \quad t \geq 0.$
- Asymptotic derivation:

$$\varepsilon := \frac{H}{L}, \ \delta := \frac{A}{H}.$$

$$(3) \qquad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = \operatorname{div}(\sigma_T) + \mathbf{g} & \text{in } \Omega_t, \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega_t, \\ \mathbf{u} = \mathbf{u}_0 & \text{in } \Omega_0. \end{cases}$$

- Incompressible fluid,
- $\mathbf{u} = (u, w)^\top$ denotes its velocity,
- $\sigma_T = -p\mathbb{I} + \mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right)$ is the total stress tensor, p denotes the pressure,
- gravity $\mathbf{g} = (0, -g)^\top$, atmospheric pressure p_0, viscosity μ and density are constants.

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Change of variables

$$x' = \frac{x}{L},\ z' = \frac{z}{H},\ t' = \frac{C_0}{L}t,$$

and

$$u' = \frac{u}{\delta C_0},\ w' = \frac{w}{\delta\varepsilon C_0},\ \eta' = \frac{\eta}{A},\ z_b' = \frac{z_b}{H},$$

where $C_0 = \sqrt{gH}$. The dimensionless coefficients are given by

$$\mu' = \frac{\mu}{C_0 L},\ p' = \frac{p}{gH},\ p_a' = \frac{p_a}{gH}.$$

Depth-averaged Mass Equation

Due to the Leibnitz integral rule and the boundary conditions, integrating the mass equation gives

$$\int_{-z_b}^{\delta\eta} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) dz = 0$$

$$\frac{\partial}{\partial x} \left(\int_{-z_b}^{\delta\eta} u dz \right) - \delta u(x, \delta\eta, t) \frac{\partial \eta}{\partial x} - u(x, -z_b, t) \frac{\partial z_b}{\partial x} + w(x, \delta\eta, t) - w(x, -z_b, t) = 0$$

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(4) $$\begin{cases} -\delta u \dfrac{\partial \eta}{\partial x} + w = \dfrac{\partial \eta}{\partial t}\sqrt{1 + (\varepsilon\delta)^2 \left|\dfrac{\partial \eta}{\partial x}\right|^2} & \text{on } (x, \delta\eta(x,t), t), \\ u\dfrac{\partial z_b}{\partial x} + w = 0 & \text{on } (x, -z_b(x), t). \end{cases}$$

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$$\frac{\partial}{\partial x} \left(\int_{-z_b}^{\delta\eta} u dz \right) - \delta u(x, \delta\eta, t) \frac{\partial \eta}{\partial x} - u(x, -z_b, t) \frac{\partial z_b}{\partial x} + w(x, \delta\eta, t) - w(x, -z_b, t) = 0$$

$$\frac{\partial (h_\delta \overline{u})}{\partial x} + \frac{\partial \eta}{\partial t} \sqrt{1 + (\varepsilon\delta)^2 \left| \frac{\partial \eta}{\partial x} \right|^2} = 0$$

We denote the depth-averaged velocity by

$$\overline{u}(x, t) = \frac{1}{h_\delta(x, t)} \int_{-z_b}^{\delta\eta} u(x, z, t) dz,$$

where $h_\delta = \delta\eta + z_b$.

The Momentum Equation (in w) yields

(4) $$\varepsilon^2\delta\left(\frac{\partial w}{\partial t}+\delta\left(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)\right)=-\frac{\partial p}{\partial z}-1+\delta\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial z}+\varepsilon^2\frac{\partial w}{\partial x}\right)\right)+2\delta\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right).$$

Hydrostatic pressure

We assume a small viscosity coefficient $\mu = \varepsilon\mu_0$.

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After rearranging terms of order ε^2 and integrating in z, we get

$$p(x,z,t) = \mathcal{O}(\varepsilon^2\delta) + (\delta\eta - z) + \varepsilon\delta\mu_0\left(\frac{\partial u}{\partial x} + 2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}(x,\delta\eta,t)\right) + p(x,\delta\eta,t) - 2\varepsilon\delta\mu_0\frac{\partial w}{\partial z}(x,\delta\eta,t)$$

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The Momentum Equation (in u) yields

$$\frac{\partial u}{\partial t} + \delta \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{1}{\delta} \frac{\partial p}{\partial x} + 2 \frac{\partial}{\partial x} \left(\varepsilon \mu_0 \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial z} \left(\varepsilon \mu_0 \left(\frac{1}{\varepsilon^2} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right).$$

Depth-averaged Momentum Equation

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Since $\mathrm{div}\,(\mathbf{u}) = 0$, integrating its left-hand side gives

$$\int_{-z_b}^{\delta\eta} \left[\frac{\partial u}{\partial t} + \delta \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) \right] dz = \int_{-z_b}^{\delta\eta} \frac{\partial u}{\partial t} dz + \delta \int_{-z_b}^{\delta\eta} \left(\frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} \right) dz$$

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$$\begin{aligned}\int_{-z_b}^{\delta\eta}\left[\frac{\partial u}{\partial t} + \delta\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right)\right]dz &= \int_{-z_b}^{\delta\eta}\frac{\partial u}{\partial t}dz + \delta\int_{-z_b}^{\delta\eta}\left(\frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z}\right)dz \\ &= \frac{\partial(h_\delta\overline{u})}{\partial t} + \delta\frac{\partial(h_\delta\overline{u^2})}{\partial x} \\ &\quad + \delta u(x,\delta\eta,t)\frac{\partial\eta}{\partial t}\left(\sqrt{1+(\varepsilon\delta)^2\left|\frac{\partial\eta}{\partial x}\right|^2} - 1\right).\end{aligned}$$

Depth-averaged Momentum Equation

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Proposition

The hydrostatic pressure, combined with the stress boundary conditions, implies that $u(x,z,t) = \overline{u}(x,t) + \mathcal{O}(\varepsilon)$. In particular, we have the approximation

$$h_\delta\overline{u^2} = h_\delta\overline{u}^2 + \mathcal{O}(\varepsilon^2).$$

To treat the right-hand side, we use the hydrostatic pressure

$$\int_{-z_b}^{\delta\eta} \left[-\frac{1}{\delta}\frac{\partial p}{\partial x} + \varepsilon\mu_0 \left(2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)\right) + \frac{\mu_0}{\varepsilon}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right] dz$$
$$= -h_\delta \frac{\partial \eta}{\partial x} + \mathcal{O}(\varepsilon) + \left[\frac{\mu_0}{\varepsilon}\frac{\partial u}{\partial z}(x,\delta\eta,t) - \frac{\mu_0}{\varepsilon}\frac{\partial u}{\partial z}(x,-z_b,t)\right].$$

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In summary, we have

$$\frac{\partial (h_\delta \overline{u})}{\partial t} + \delta \frac{\partial (h_\delta \overline{u}^2)}{\partial x} = -h_\delta \frac{\partial \eta}{\partial x} + \mathcal{O}(\varepsilon)$$
$$+ \left[\frac{\mu_0}{\varepsilon}\frac{\partial u}{\partial z}(x,\delta\eta,t) - \frac{\mu_0}{\varepsilon}\frac{\partial u}{\partial z}(x,-z_b,t)\right]$$
$$+ \delta u(x,\delta\eta,t)\frac{\partial \eta}{\partial t}\left(\sqrt{1+(\varepsilon\delta)^2\left|\frac{\partial \eta}{\partial x}\right|^2} - 1\right).$$

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$$\int_{-z_b}^{\delta\eta}\left[-\frac{1}{\delta}\frac{\partial p}{\partial x}+\varepsilon\mu_0\left(2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)\right)+\frac{\mu_0}{\varepsilon}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)\right]dz$$
$$=-h_\delta\frac{\partial \eta}{\partial x}+\mathcal{O}(\varepsilon)+\left[\frac{\mu_0}{\varepsilon}\frac{\partial u}{\partial z}(x,\delta\eta,t)-\frac{\mu_0}{\varepsilon}\frac{\partial u}{\partial z}(x,-z_b,t)\right].$$

In summary, we have

$$\frac{\partial(h_\delta\overline{u})}{\partial x}+\frac{\partial\eta}{\partial t}\sqrt{1+(\varepsilon\delta)^2\left|\frac{\partial\eta}{\partial x}\right|^2}=0,$$
$$\frac{\partial(h_\delta\overline{u})}{\partial t}+\delta\frac{\partial(h_\delta\overline{u}^2)}{\partial x}=-h_\delta\frac{\partial\eta}{\partial x}+\mathcal{O}(\varepsilon)$$
$$+\left[\frac{\mu_0}{\varepsilon}\frac{\partial u}{\partial z}(x,\delta\eta,t)-\frac{\mu_0}{\varepsilon}\frac{\partial u}{\partial z}(x,-z_b,t)\right]$$
$$+\delta u(x,\delta\eta,t)\frac{\partial\eta}{\partial t}\left(\sqrt{1+(\varepsilon\delta)^2\left|\frac{\partial\eta}{\partial x}\right|^2}-1\right).$$

Navier-Stokes

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = \operatorname{div}(\sigma_T) + \mathbf{g} & \text{in } \Omega_t, \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega_t, \\ \mathbf{u} = \mathbf{u}_0 & \text{in } \Omega_0. \end{cases}$$

$$\varepsilon = \frac{H}{L} \longrightarrow 0$$

(Long wave theory)

Saint-Venant

$$\begin{cases} \dfrac{\partial h\overline{u}}{\partial x} + \dfrac{\partial \eta}{\partial t} = 0, \\ \dfrac{\partial h\overline{u}}{\partial t} + \dfrac{\partial h\overline{u}^2}{\partial x} + gh\dfrac{\partial \eta}{\partial x} = 0. \end{cases}$$

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$$\delta = \frac{A}{H} \longrightarrow 0$$

(Small amplitude theory)

Wave Equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(g z_b \frac{\partial \eta}{\partial x}\right) = 0.$$

Helmholtz formulation

In a two-dimensional setting, $\eta(x,t) = \mathrm{Re}\{\psi_{tot}(x)\mathrm{e}^{-\mathrm{i}\omega t}\}$ is a solution of the Wave Equation, where the amplitude ψ_{tot} satisfies

$$\omega^2\psi_{tot} + \operatorname{div}(gz_b\nabla\psi_{tot}) = 0. \tag{5}$$

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The bathymetry can be decomposed as $z_b(x) := z_0 + \delta z_b(x)$, with z_0 constant and δz_b has compact support in Ω.

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We then reformulate (5) as

Total wave

$$\begin{cases} \operatorname{div}\left((1+q)\nabla \psi_{tot}\right) + k_0^2 \psi_{tot} = 0 & \text{in } \Omega, \\ \nabla(\psi_{tot} - \psi_0)\cdot \hat{n} - \mathrm{i}k_0(\psi_{tot} - \psi_0) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $q(x) := \frac{\delta z_b(x)}{z_0}$ is compactly supported in Ω, $k_0 := \frac{\omega}{\sqrt{g z_0}}$, $\hat{n}$ is the unit normal to $\partial\Omega$ and $\psi_0(x) = \mathrm{e}^{\mathrm{i}k_0 x\cdot\vec{d}}$ (s.t. $|\vec{d}| = 1$).

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Scattered wave ($\psi_{tot} = \psi_0 + \psi_{sc}$)

$$\begin{cases} \operatorname{div}\left((1+q)\nabla\psi_{sc}\right) + k_0^2 \psi_{sc} = -\operatorname{div}\left(q \nabla \psi_0\right) & \text{in } \Omega, \\ \nabla\psi_{sc}\cdot\hat{n} - \mathrm{i}k_0\psi_{sc} = 0 & \text{on } \partial\Omega. \end{cases}$$

where $q(x) := \frac{\delta z_b(x)}{z_0}$ is compactly supported in Ω, $k_0 := \frac{\omega}{\sqrt{g z_0}}$, $\hat{n}$ is the unit normal to $\partial\Omega$ and $\psi_0(x) = \mathrm{e}^{\mathrm{i}k_0 x\cdot\vec{d}}$ (s.t. $|\vec{d}| = 1$).

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In a two-dimensional setting, $\eta(x,t) = \mathrm{Re}\{\psi_{tot}(x)\mathrm{e}^{-\mathrm{i}\omega t}\}$ is a solution of the Wave Equation, where the amplitude ψ_{tot} satisfies

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The bathymetry can be decomposed as $z_b(x) := z_0 + \delta z_b(x)$, with z_0 constant and δz_b has compact support in Ω.

We then consider the following problem

Helmholtz formulation

(6) $$\begin{cases} -\mathrm{div}\,((1+q)\nabla\psi) - k_0^2\psi = \mathrm{div}\,(q\nabla\psi_0) & \text{in } \Omega, \\ (1+q)\nabla\psi\cdot\hat{n} - \mathrm{i}k_0\psi = g - q\nabla\psi_0\cdot\hat{n} & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open set with Lipschitz boundary, $q \in L^\infty(\Omega)$ satisfying

(7) $$(\exists\alpha > 0) \quad 1 + q(x) \geq \alpha \quad \text{a.e. } x \in \Omega.$$

A weak formulation for (6) is given by

$$a(q;\psi,\phi) = b(q;\phi) \qquad \forall \phi \in H^1(\Omega), \tag{8}$$

where

$$a(q;\psi,\phi) := \int_\Omega \left((1+q)\nabla\psi \cdot \nabla\overline{\phi} - k_0^2\psi\overline{\phi}\right)\, dx - \mathrm{i}k_0 \int_{\partial\Omega} \psi\overline{\phi}\, d\sigma,$$

$$b(q;\phi) := -\int_\Omega q\nabla\psi_0 \cdot \nabla\overline{\phi}\, dx + \langle g, \overline{\phi}\rangle_{H^{-1/2},H^{1/2}}.$$

Weak formulation

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$$b(q;\phi) := -\int_\Omega q\nabla\psi_0\cdot\nabla\overline{\phi}\,dx + \langle g,\overline{\phi}\rangle_{H^{-1/2},H^{1/2}}.$$

The sesquilinear form a:

- is continuous under the norm

$$\|\psi\|_{1,k_0}^2 := k_0^2\,\|\psi\|_{L^2(\Omega)}^2 + \alpha\,\|\nabla\psi\|_{L^2(\Omega)}^2\,.$$

- Satisfies a Gårding inequality

$$\mathrm{Re}\{a(q;\psi,\psi)\} + 2k_0^2\,\|\psi\|_{L^2(\Omega)}^2 \geq \|\psi\|_{1,k_0}^2\,.$$

We are interested in solving the next PDE-constrained optimization problem

$$\begin{aligned} \min_{(q,\psi)\in U_\Lambda \times H^1(\Omega)} & J(q,\psi) \\ \text{s.t.} \qquad & (8). \end{aligned} \tag{9}$$

where $U_\Lambda = \{q \in BV(\Omega) \mid \alpha - 1 \leq q(x) \leq \Lambda \; a.e. \; x \in \Omega\}$ is a closed, weakly* closed and convex subset of $BV(\Omega)$.

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Space of functions of Bounded Variations

- Banach space for the norm $\|q\|_{BV(\Omega)} := \|q\|_{L^1(\Omega)} + |Dq|(\Omega)$, where D is the distributional gradient and $|Dq|(\Omega)$ is the variation of q.
- The weak* convergence means

$$q_n \to q \text{ in } L^1(\Omega) \text{ and } Dq_n \rightharpoonup Dq \text{ in } \mathcal{M}_\mathrm{b}(\Omega, \mathbb{R}^N).$$

- The application $q \in BV(\Omega) \mapsto |Dq|(\Omega) \in \mathbb{R}^+$ is lower semi-continuous with respect to the weak* topology of BV.

Continuity of the control-to-state mapping

Theorem

Assume that $q \in U_\Lambda$. Then there exists a constant $C_\mathrm{s}(k_0, \Omega) > 0$ such that

$$\|\psi\|_{1,k_0} \leq C_\mathrm{s}(k_0, \Omega) \sup_{\|\phi\|_{1,k_0}=1} |a(q; \psi, \phi)|.$$

In addition, the solution to (8) satisfies the bound

$$\begin{aligned} \|\psi\|_{1,k_0} \leq C_\mathrm{s}(k_0, \Omega) C(\Omega) \max\{k_0^{-1}, \alpha^{-1/2}\} \\ \times \left(\|q\|_{L^\infty(\Omega)} \|\nabla \psi_0\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right), \end{aligned}$$

with $C(\Omega) > 0$.

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with $C(\Omega) > 0$.

As a result of this theorem and the continuity of the trace, we have

$$\|\psi_{tot}\|_{1,k_0} \leq C(\Omega) C_{\mathrm{s}}(k_0, \Omega) k_0 \max\{k_0^{-1}, \alpha^{-1/2}\},$$
$$\|\psi_{sc}\|_{1,k_0} \leq k_0 C_{\mathrm{s}}(k_0, \Omega) \alpha^{-1/2} \|q\|_{L^\infty(\Omega)} \sqrt{|\Omega|}.$$

Theorem

Let $(q_n)_n \subset U$ be a sequence satisfying $\|q_n\|_{BV(\Omega)} \leq M$ and whose weak* limit in $BV(\Omega)$ is denoted by q_∞. Let $(\psi(q_n))_n$ be the sequence of weak solution to Problem (8). Then $\psi(q_n)$ converges strongly in $H^1(\Omega)$ towards $\psi(q_\infty)$. In other words, the mapping

$$q \in (U_\Lambda, \text{weak}^*) \mapsto \psi(q) \in (H^1(\Omega), \text{strong}),$$

is continuous.

Theorem ▶ Existence of optimal solution [Cocquet, Riffo and Salomon]

Assume that the cost function $(q, \psi) \in U_\Lambda \mapsto J(q, \psi) \in \mathbb{R}$ satisfies:

(A1) There exists $\beta > 0$ such that

$$J(q, \psi) = J_0(q, \psi) + \beta |Dq|(\Omega).$$

(A2) $\forall (q, \psi) \in U_\Lambda \times H^1(\Omega)$, $J_0(q, \psi) \geq m > -\infty$.

(A3) $(q, \psi) \mapsto J_0(q, \psi)$ is lower-semi-continuous with respect to the (weak*,weak) topology of $BV(\Omega) \times H^1(\Omega)$.

Then the optimization problem (9) has at least one optimal solution in $U_\Lambda \times H^1(\Omega)$.

Boundedness/Continuity of solution to Helmholtz problem

Theorem

Assume that $q \in L^\infty(\Omega)$ and satisfies (7) and $g \in L^2(\partial\Omega)$. Then the solution to Problem (8) satisfies

$$\|\psi\|_{C^0(\Omega)} \leq \widetilde{C}(\Omega)\widetilde{C}_{\mathrm{s}}(k_0,\alpha)\left(\|q\|_{L^\infty(\Omega)}\|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right),$$

where $\widetilde{C}(\Omega) > 0$ and

$$\widetilde{C}_{\mathrm{s}}(k_0,\alpha) = 1 + \left((1+k_0^2)k_0^{-1} + \alpha^{-1/2}\right)\max\{k_0^{-1},\alpha^{-1/2}\}C_{\mathrm{s}}(k_0,\Omega).$$

Boundedness/Continuity of solution to Helmholtz problem

Theorem

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Consequently,

$$\|\psi_{tot}\|_{\mathcal{C}^0(\Omega)} \leq k_0\widetilde{C}(\Omega)\widetilde{C}_{\mathrm{s}}(k_0,\alpha),$$

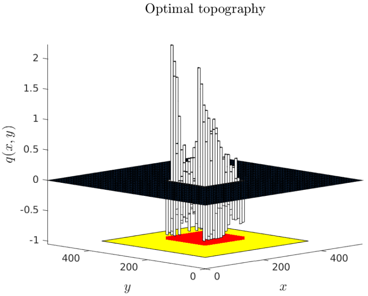
$$\|\psi_{sc}\|_{\mathcal{C}^0(\Omega)} \leq k_0\widetilde{C}(\Omega)\left[\left((1+k_0^2)k_0^{-1} + \alpha^{-1/2}\right)\alpha^{-1/2}C_{\mathrm{s}}(k_0,\Omega) + 1\right]\|q\|_{L^\infty(\Omega)}.$$

Minimization of the cost functional

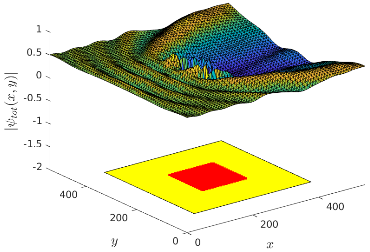
$$J(q, \psi_{tot}) = \frac{\omega_0^2}{2} \int\limits_{\Omega_0 = [\frac{L}{6}, \frac{5L}{6}]^2} |\psi_{tot}(x, y)|^2 dx dy,$$

The bathymetry is only optimized on a subset $\Omega_q = [\frac{L}{4}, \frac{3L}{4}]^2 \subset \Omega_0$.

Optimal bathymetry for a wave damping problem



(a) View from above.



(b) Norm of the numerical solution.

Minimization of the cost functional

$$J(q,\psi_{tot}) = \frac{\omega_0^2}{2} \int_{\Omega_0 = [\frac{L}{6}, \frac{5L}{6}]^2} |\psi_{tot}(x,y)|^2 dxdy,$$

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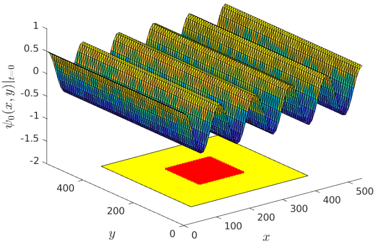
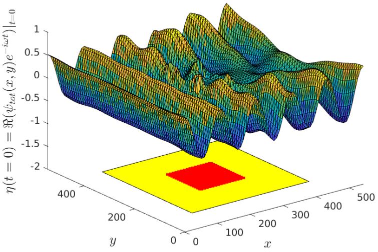


(a) Real part of the incident wave.

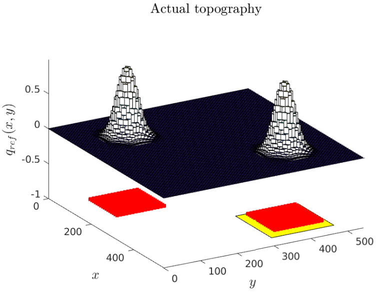


(b) Real part of the numerical solution.

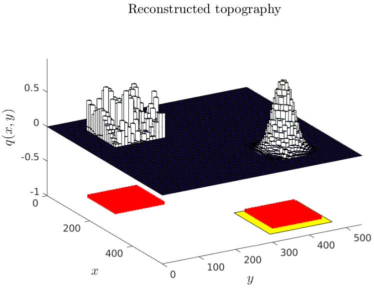
Minimization of the cost functional

$$J(q, \psi_{tot}) = \frac{\omega_0^2}{2} \int_{\Omega_0 = [\frac{3L}{4} - \delta, \frac{3L}{4} + \delta]^2} |\psi_{tot}(x, y) - \psi_{ref}(x, y)|^2 dx dy,$$

where ψ_{ref} is the amplitude associated with q_{ref}, $\delta = \frac{L}{6}$ and $\Omega_q = [\frac{L}{8}, \frac{3L}{8}]^2 \cup [\frac{5L}{8}, \frac{7L}{8}]^2$.



(a) Actual bathymetry.



(b) Reconstructed bathymetry.

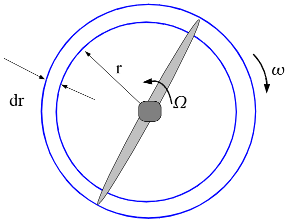
Minimization of the cost functional

$$J(q,\psi_{tot}) = \frac{\omega_0^2}{2} \int_{\Omega_0 = [\frac{3L}{4}-\delta,\, \frac{3L}{4}+\delta]^2} |\psi_{tot}(x,y) - \psi_{ref}(x,y)|^2 dxdy,$$

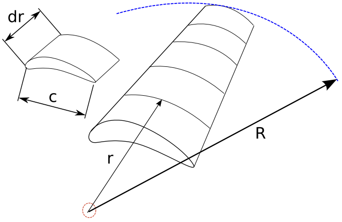
where ψ_{ref} is the amplitude associated with q_{ref}, $\delta = \frac{L}{6}$ and $\Omega_q = [\frac{L}{8}, \frac{3L}{8}]^2 \cup [\frac{5L}{8}, \frac{7L}{8}]^2$.

- A natural extension is to consider a polychromatic wave.
- Several questions have to be addressed in between, regarding first a possible decomposition of the cost functional and then the convergence of the whole procedure.
- This idea cannot be extended to nonlinear wave propagation models as Saint-Venant or Boussinesq.

Outline

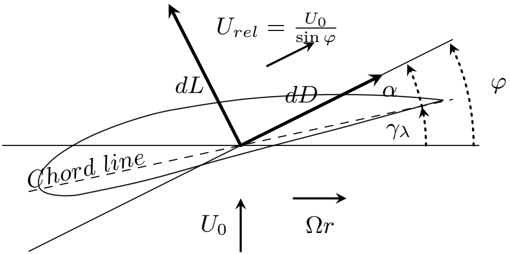


(a) Radial decomposition



(b) Blade element model

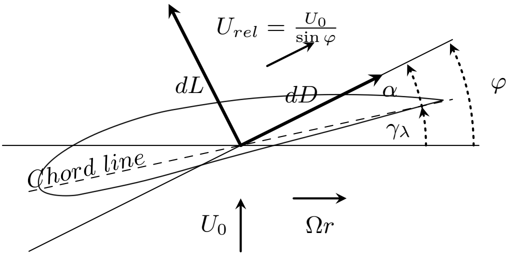
$$
\begin{aligned}
a &= \frac{U_{-\infty} - U_0}{U_{-\infty}}, \\
a' &= \frac{\omega}{2\Omega}, \\
\tan\varphi &= \frac{1-a}{\lambda(1+a')}.
\end{aligned}
$$



Blade element profile and associated angles, velocities and forces.

$$dL = C_L(\alpha)\frac{\rho}{2}U_{rel}^2 c_\lambda dr,$$

$$dD = C_D(\alpha)\frac{\rho}{2}U_{rel}^2 c_\lambda dr.$$



Blade element profile and associated angles, velocities and forces.

$$dL = C_L(\alpha)\frac{\rho}{2}U_{rel}^2 c_\lambda dr,$$
$$dD = C_D(\alpha)\frac{\rho}{2}U_{rel}^2 c_\lambda dr.$$

Assumption 3.1

In what follows, we assume that C_L is well-defined and continuous on an interval

$$I_\beta := [-\beta, \beta]$$

for some $\beta \in [0, \alpha_s)$ and positive on $I_\beta \cap \mathbb{R}^+$. The coefficient C_D is well-defined and positive on $\mathbb{R}$.

We denote by dT the infinitesimal thrust and dQ the infinitesimal torque that apply on the blade element under consideration.

Macroscopic approach (Momentum Theory)

$$dT = 4a(1-a)U_{-\infty}^2 \rho\pi r dr,$$
$$dQ = 4a'(1-a)\lambda U_{-\infty}^2 \rho\pi r^2 dr.$$

Local expressions (Blade Element Theory)

$$dT = \sigma_\lambda \frac{(1-a)^2}{\sin^2\varphi}\left(C_L(\varphi-\gamma_\lambda)\cos\varphi + C_D(\varphi-\gamma_\lambda)\sin\varphi\right)U_{-\infty}^2 \rho\pi r dr,$$
$$dQ = \sigma_\lambda \frac{(1-a)^2}{\sin^2\varphi}\left(C_L(\varphi-\gamma_\lambda)\sin\varphi - C_D(\varphi-\gamma_\lambda)\cos\varphi\right)U_{-\infty}^2 \rho\pi r^2 dr,$$

with $\sigma_\lambda = \frac{Bc_\lambda}{2\pi r}$.

Glauert's relations

$$
\begin{aligned}
\tan\varphi &= \frac{1-a}{\lambda(1+a')},\\
\frac{a}{1-a} &= \frac{\sigma_\lambda}{4\sin^2\varphi}\left(C_L(\varphi-\gamma_\lambda)\cos\varphi + C_D(\varphi-\gamma_\lambda)\sin\varphi\right),\\
\frac{a'}{1-a} &= \frac{\sigma_\lambda}{4\lambda\sin^2\varphi}\left(C_L(\varphi-\gamma_\lambda)\sin\varphi - C_D(\varphi-\gamma_\lambda)\cos\varphi\right).
\end{aligned}
$$

Simplified model ($C_D = 0$)

Glauert's relations become

(10) $$\tan\varphi = \frac{1-a}{\lambda(1+a')},$$

(11) $$\frac{a}{1-a} = \mu_L(\varphi)\frac{\cos\varphi}{\sin^2\varphi},$$

(12) $$\frac{a'}{1-a} = \mu_L(\varphi)\frac{1}{\lambda\sin\varphi},$$

with $\mu_L(\varphi) := \frac{\sigma_\lambda}{4}C_L(\varphi-\gamma_\lambda)$ defined on $I_{\beta,\gamma_\lambda} := [-\beta+\gamma_\lambda, \beta+\gamma_\lambda]$.

Simplified model ($C_D = 0$)

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with $\mu_L(\varphi) := \frac{\sigma_\lambda}{4}C_L(\varphi-\gamma_\lambda)$ defined on $I_{\beta,\gamma_\lambda} := [-\beta+\gamma_\lambda, \beta+\gamma_\lambda]$.

Theorem 3.2 ▶ Reformulation of the simplified model

Suppose that Assumption 3.1 holds and that $(\varphi, a, a') \in I - \{0, \frac{\pi}{2}\} \times \mathbb{R} - \{1\} \times \mathbb{R} - \{-1\}$ satisfies Eqs (10–12). Then φ satisfies

(13) $$\mu_L(\varphi) = \mu_G(\varphi),$$

where $\mu_G(\varphi) := \sin\varphi\tan(\theta_\lambda - \varphi)$. Reciprocally, suppose that $\varphi \in I - \{0, \frac{\pi}{2}\}$ satisfies Eq. (13) and define a and a' as the corresponding solutions of Eqs. (11) and (12), respectively. Then $(\varphi, a, a') \in I - \{0, \frac{\pi}{2}\} \times \mathbb{R} - \{1\} \times \mathbb{R} - \{-1\}$ satisfies Eqs. (10–12).

Corrected model

(14) $$\tan\varphi = \frac{1-a}{\lambda(1+a')},$$

(15) $$\frac{a}{1-a} = \frac{1}{\sin^2\varphi}(\mu_L^c(\varphi)\cos\varphi + \mu_D^c(\varphi)\sin\varphi) - \frac{\psi\left((a-a_c)_+\right)}{(1-a)^2},$$

(16) $$\frac{a'}{1-a} = \frac{1}{\lambda\sin^2\varphi}(\mu_L^c(\varphi)\sin\varphi - \mu_D^c(\varphi)\cos\varphi),$$

where $\mu_L^c(\varphi) := \frac{\sigma_\lambda}{4F_\lambda(\varphi)}C_L(\varphi-\gamma_\lambda)$, $\mu_D^c(\varphi) := \frac{\sigma_\lambda}{4F_\lambda(\varphi)}C_D(\varphi-\gamma_\lambda)$, defined respectively on I_{β,γ_λ} and $\mathbb{R}$.

Corrected model

$$\text{(14)} \qquad \tan\varphi = \frac{1-a}{\lambda(1+a')},$$

$$\text{(15)} \qquad \frac{a}{1-a} = \frac{1}{\sin^2\varphi}(\mu_L^c(\varphi)\cos\varphi + \mu_D^c(\varphi)\sin\varphi) - \frac{\psi\left((a-a_c)_+\right)}{(1-a)^2},$$

$$\text{(16)} \qquad \frac{a'}{1-a} = \frac{1}{\lambda\sin^2\varphi}(\mu_L^c(\varphi)\sin\varphi - \mu_D^c(\varphi)\cos\varphi),$$

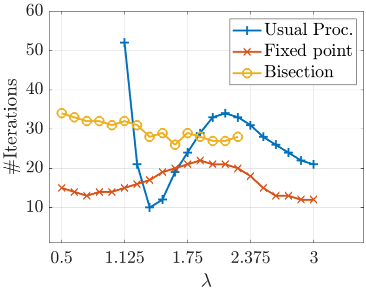
where $\mu_L^c(\varphi) := \frac{\sigma\lambda}{4F_\lambda(\varphi)}C_L(\varphi-\gamma_\lambda)$, $\mu_D^c(\varphi) := \frac{\sigma\lambda}{4F_\lambda(\varphi)}C_D(\varphi-\gamma_\lambda)$, defined respectively on I_{β,γ_λ} and $\mathbb{R}$.

Theorem 3.3 ▶ Reformulation of the corrected model

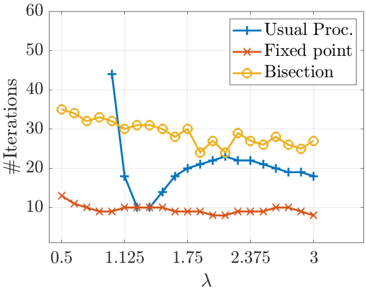
(...) Suppose also that $(\varphi, a, a') \in I^+ - \{\frac{\pi}{2}\} \times \mathbb{R} - \{1\} \times \mathbb{R}$ satisfies Equations (14–16). Then φ satisfies

$$\text{(17)} \qquad \mu_L^c(\varphi) - \tan(\theta_\lambda - \varphi)\mu_D^c(\varphi) = \mu_G^c(\varphi),$$

where $\mu_G^c(\varphi) := \mu_G(\varphi) + \dfrac{\cos\theta_\lambda \sin^2\varphi}{\cos(\theta_\lambda-\varphi)} \dfrac{\psi\left((\tau(\varphi)-a_c)_+\right)}{(1-\tau(\varphi))^2}$. (...)



(a) $a_c = 0.2$



(b) $a_c = 1$

Number of iterations of usual (Usual Proc.), fixed-point (Fixed point) and bisection (Bisection) algorithms required to solve Equation (17), according to the criterium

$$\left| \mu_L^c(\varphi^k) - \tan(\theta_\lambda - \varphi^k)\mu_D^c(\varphi^k) - \mu_G^c(\varphi^k) \right| \leq \text{Tol} = 10^{-10}$$

and for various values of λ.

The design procedure mainly consists in optimizing the power coefficient

$$\max\ C_p(\gamma_\lambda, c_\lambda, \varphi) = \frac{8}{\lambda_{\max}^2} \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^3 a'(1-a) \left(1 - \frac{C_D}{C_L} \tan^{-1}\varphi\right) d\lambda,$$

$$\text{s.t.} \begin{cases} \tan\varphi = \dfrac{1-a}{\lambda(1+a')}, \\ \dfrac{a}{1-a} = \dfrac{1}{\sin^2\varphi}(\mu_L^c(\varphi)\cos\varphi + \mu_D^c(\varphi)\sin\varphi) - \dfrac{\psi\left((a-a_c)_+\right)}{(1-a)^2}, \\ \dfrac{a'}{1-a} = \dfrac{1}{\lambda\sin^2\varphi}(\mu_L^c(\varphi)\sin\varphi - \mu_D^c(\varphi)\cos\varphi). \end{cases}$$

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Several simplifications are taken into account:

- Assume $F_\lambda(\varphi) = 1$ and $\psi\left((a - a_c)_+\right) = 0$.

The design procedure mainly consists in optimizing the power coefficient

$$\max \; C_p(\cancel{\gamma_\lambda}, c_\lambda, \varphi) = \frac{8}{\lambda_{\max}^2} \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^3 a'(1-a) \left(1 - \frac{C_D}{C_L} \tan^{-1} \varphi\right) d\lambda,$$

$$\text{s.t.} \begin{cases} \tan\varphi = \dfrac{1-a}{\lambda(1+a')}, \\ \dfrac{a}{1-a} = \dfrac{1}{\sin^2\varphi}(\mu_L(\varphi)\cos\varphi + \mu_D^c(\varphi)\sin\varphi), \\ \dfrac{a'}{1-a} = \dfrac{1}{\lambda\sin^2\varphi}(\mu_L(\varphi)\sin\varphi - \mu_D^c(\varphi)\cos\varphi). \end{cases}$$

Several simplifications are taken into account:

- Assume $F_\lambda(\varphi) = 1$ and $\psi\left((a - a_c)_+\right) = 0$.
- Define $\overline{\alpha} = \varphi - \gamma_\lambda$ that minimizes $\frac{C_D}{C_L}$.

Optimization

The design procedure mainly consists in optimizing the power coefficient

$$\max \; C_p(\cancel{\gamma_\lambda}, c_\lambda, \varphi) = \frac{8}{\lambda_{\max}^2} \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^3 a'(1-a) d\lambda,$$

$$\text{s.t.} \begin{cases} \tan\varphi = \dfrac{1-a}{\lambda(1+a')}, \\ \dfrac{a}{1-a} = \dfrac{1}{\sin^2\varphi}\mu_L(\varphi)\cos\varphi, \\ \dfrac{a'}{1-a} = \dfrac{1}{\lambda\sin^2\varphi}\mu_L(\varphi)\sin\varphi. \end{cases}$$

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- Then the coefficient C_D is simply neglected.

Optimization

The design procedure mainly consists in optimizing the power coefficient

$$\max\ C_p(\cancel{\gamma_\lambda}, \cancel{c_\lambda}, \varphi) = \frac{8}{\lambda_{\max}^2}\int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^3 a'(1-a)d\lambda,$$

$$\text{s.t.}\begin{cases} \tan\varphi = \dfrac{1-a}{\lambda(1+a')}, \\ \dfrac{a}{1-a} = \dfrac{1}{\sin^2\varphi}\mu_L(\varphi)\cos\varphi, \\ \dfrac{a'}{1-a} = \dfrac{1}{\lambda\sin^2\varphi}\mu_L(\varphi)\sin\varphi. \end{cases}$$

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- Then the coefficient C_D is simply neglected.
- Theorem 3.2 allows us to replace $\mu_L(\varphi) := \frac{\sigma_\lambda}{4}C_L(\varphi-\gamma_\lambda)$ by $\mu_G(\varphi)$.

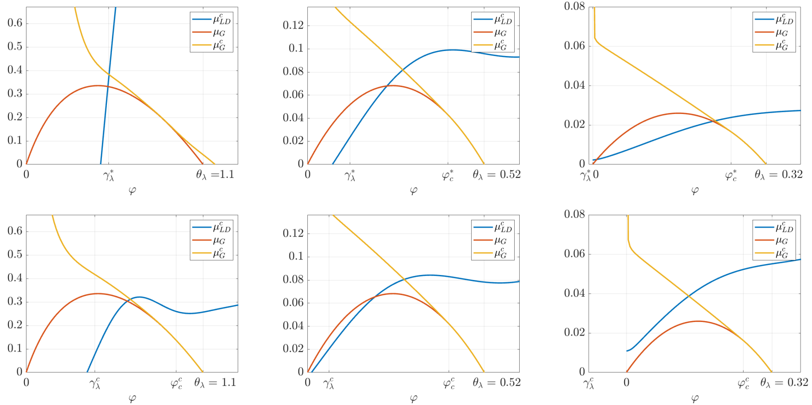
The design procedure mainly consists in optimizing the power coefficient

$$\max_{\varphi} \quad C_p(\varphi) \ = \frac{8}{\lambda_{\max}} \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^3 a'(1-a) d\lambda$$

$$\text{s.t.} \begin{cases} \tan\varphi = \dfrac{1-a}{\lambda(1+a')} \\ \dfrac{a}{1-a} = \mu_G(\varphi)\dfrac{\cos\varphi}{\sin^2\varphi} \\ \dfrac{a'}{1-a} = \mu_G(\varphi)\dfrac{1}{\lambda\sin\varphi}. \end{cases}$$

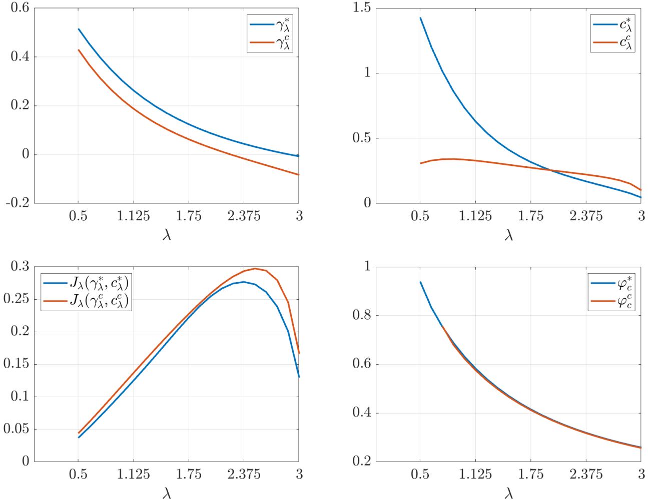
Several simplifications are taken into account:

- Assume $F_\lambda(\varphi) = 1$ and $\psi\left((a-a_c)_+\right) = 0$.
- Define $\overline{\alpha} = \varphi - \gamma_\lambda$ that minimizes $\frac{C_D}{C_L}$.
- Then the coefficient C_D is simply neglected.
- Theorem 3.2 allows us to replace $\mu_L(\varphi) := \frac{\sigma_\lambda}{4} C_L(\varphi - \gamma_\lambda)$ by $\mu_G(\varphi)$.
- The solution of the new problem is $\gamma_\lambda^* = \varphi^* - \overline{\alpha}, \ \ c_\lambda^* = \dfrac{8\pi r \mu_G(\varphi^*)}{BC_L(\overline{\alpha})}$.



(a) $\lambda_1 = 0.5$ (b) $\lambda_2 = 1.75$ (c) $\lambda_3 = 3$

Graphs of the functions $\mu_{LD}^c : \varphi \mapsto \mu_L^c(\varphi) - \tan(\theta_\lambda - \varphi)\mu_D^c(\varphi)$, μ_G^c and μ_G for various values of λ.
Above: $(\gamma_\lambda, c_\lambda) = (\gamma_\lambda^*, c_\lambda^*)$, below: $(\gamma_\lambda, c_\lambda) = (\gamma_\lambda^c, c_\lambda^c)$.



Graphs of the functions γ_λ^* and γ_λ^c, c_λ^* and c_λ^c, $J_\lambda(\gamma_\lambda^*, c_\lambda^*)$ and $J_\lambda(\gamma_\lambda^c, c_\lambda^c)$ and the corresponding φ_c.

- Concerning the convergence of solving algorithms, extending the proof to a more general framework is desirable.
- An asymptotic analysis gives a general idea of the optimal solution behavior, however the required assumptions seem very restrictive.
- The question of multiple optima in the corrected model remains open.

Thank you for your attention !